① Airy structures

V vector space / $\mathbb{C}$ — over $\mathbb{C}$ the base of V; over the ring of functions on a manifold

$V[\![\hbar]\!]$ = $\{(\omega_0, \omega_1, \omega_2, \ldots) \mid \omega_g \in \mathrm{Sym}^{2g}(V^*)\}$ — formal power series

$\deg \hbar = 2$ ⟶ $\deg \omega_g = g$

Def A quantum Airy structure (QAS) on

i. $V \longrightarrow V[\![\hbar]\!]$ satisfying

ii. $\deg \omega_g = g$ (equivalently $L(\hbar) = \frac{\hbar}{2}\omega_0(\frac{\partial}{\partial \hbar}, \cdot) + O(\hbar^2)$)

iii. $\mathrm{Im}\, L(\hbar) \subset V[\![\hbar]\!]$ is a Lie subalgebra, $[L_1, L_2] = \omega_0(L_1, L_2)$ (compatibility of brackets)

Lie bracket $\frac{1}{\hbar}\omega_0^{\flat} = 0$

simplest example: L has terms of maximal deg 2

$L = \frac{\hbar}{2}\partial_x^2 + \frac{1}{\hbar}(\frac{1}{2}Ax^2 + Bx\partial_x + \frac{1}{2}C\hbar^2\partial_x^2) - D$

$A, B, C, D \in \mathbb{C}$ scalars

$[L_1, L_2] = \hbar^2\{\{H_1, H_2\}\} \longrightarrow [L_1, L_2] = \hbar\{H_1, H_2\}$ (to a Lie algebra, from a Lie algebra)

Def+Theorem $\exists! F \in \hbar^2\mathrm{Sym}(V^*)[\![\hbar]\!]$ $\quad F = \sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(l_1, \ldots, l_n) x_{l_1} \cdots x_{l_n}$

i. $F_{0,1} L e^F \equiv 0$ $\quad \deg = g - 2 + n > 0$

ii. $F_{g,n} = 0$ $\forall g$ $\quad e^F \equiv 1 + \cdots$

$F_{0,1} = 0$, $F_{0,2} = 0$

Proof of uniqueness: in $e^F L e^F \equiv 0$, extract coeff of $\hbar^g x_{l_1} \cdots x_{l_n}$, $n > 0$

$$F_{g,n}(l_1, \ldots, l_n) - \tilde{\partial}_{l_1} F_{g,n-1}(l_2, \ldots, l_n) - \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\ldots,n\}}} F_{g_1,|I_1|+1}(l_1, I_1) F_{g_2,|I_2|+1}(l_2, I_2) - \frac{1}{2}\sum_{a,b} \omega_0^{ab}\left(F_{g-1,n+1}(a,b,\ldots) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{\ldots\}}} F_{g_1,|I_1|+1}(a, I_1) F_{g_2,|I_2|+1}(b, I_2)\right) - D^2 F_{g,n} = 0$$

$\Rightarrow F_{g,n} = \mathrm{Res}_{\infty}[F_{g,n}\, \cdots] \quad 2g-2+n < 2g-2+n$

$2g-1) \quad -2+(n+1) = 2g-2+n$

$< 2g-2+n$.

$F_0 = 0$

$F_1 = 0$

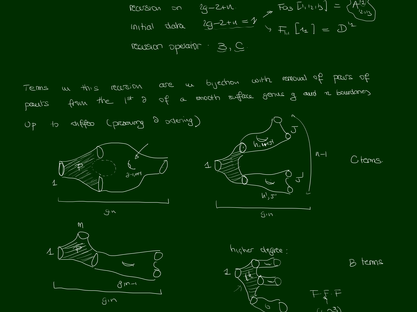
recursion on $2g-2+n$ $\quad \mathrm{Res}[\cdots] = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{2g-2+n}$

initial data $2g-2+n = 1$ $\quad \Rightarrow F_{1,1}[\psi_1] = D^2$

recursion operator: ∂_1, Δ

Terms in this recursion are in bijection with removal of pairs of pants from the 1^{st} 3 of a smooth surface genus g and n boundaries

Up to suffix (ordering & orientation)



Symmetry is not obvious from the recursion.

This system is fully symmetric.

Existence: comes from geodesics in moduli space

?? → needs adding

QAS | TR (recursion on $2g-2+n>0$) | $F_{g,n}$ (partition fn of e^F) | enumerative geometry ?

algebraic version | Eynard-Orantin topological recursion | $2g$ geometry

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- $V = \mathbb{C}$ $\quad L = \hbar\partial_x^2 - \frac{1}{2}Ax^2 - Bx\partial_x - \frac{1}{2}C\hbar^2\partial_x^2 - D$

  $A = B = C = D = \frac{1}{2}$

  $Le^F = 0$ $\quad e^F = \mathrm{Ai}\left[\frac{x - a\hbar}{\hbar^{2/3}}\right]$ $\quad \mathrm{Ai}' = x\mathrm{Ai}$

- $V = \mathbb{C}[\mathrm{Tell}]$ $\quad$ (L) i.e.: Virasoro constraints for GW(Spt) $\quad$ $\mathbb{P}^1$-descendants

  e^F = Witten-Kontsevich partition

  = Matrix Airy function

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② Milanov: $\mathcal{W}$-constraints $\Longleftrightarrow$ higher order TR

VOA + automorphism ⟶ ?? QAS

free field representation

$W(g)\mathrm{Id}_{V^{\otimes d}}$ — Miura transform

Def A VOA is $\mathbb{Z}$-graded vector space V ["state space"]

equipped with

- $1 \in V$ $\quad Y(1, z) = \mathrm{Id}_V$
  $Y(a, z)1 \in a + zV[\![z]\!]$ $\quad \forall a \in V$

- $|\omega\rangle \in V$ $\quad Y(\omega, z) = \sum_{n \in \mathbb{Z}} \frac{L_{n+1}}{z^{n+2}}$

  $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathrm{Id}_V$ for some $c \in \mathbb{C}$ (central charge)

- $Y(L_{-1}1, z) = \partial_z Y(1, z)$

- locality

If $\mathcal{W}$ with conformal vector = eigenvalue for L_0

$Y(a, z) = \sum_{n \in \mathbb{Z}} \frac{a_{(n)}}{z^{n+1}}$ modes $\in \mathrm{End}\, V$

VOA ⟶ (complex) of the (anomalous) algebra of modes.

Ⓐ Heisenberg algebra $\quad \mathfrak{g} = \mathfrak{g}\mathfrak{l}_\infty$

Cartan $\mathfrak{h} = \mathbb{C}^\infty$ ⟶ $\hat{\mathfrak{h}} = (\mathfrak{h}[[t, t^{-1}]] \oplus \mathbb{C}K)[[\hbar]]$

Killing form $\langle , \rangle$, $\xi \in \mathfrak{h}$ $\quad \hat{\xi}_n = \xi \otimes t^n$

$\hat{\mathfrak{h}}_- = \mathrm{span}(\hat{\xi}_n \mid n > 0, \xi \in \mathfrak{h})$

Lie alg structure $[\hat{\xi}_n, \eta_m] = n\delta_{n+m,0}\langle \xi, \eta \rangle$

$V = \mathrm{Sym}(\hat{\mathfrak{h}}_-).|0\rangle$ has a VOA structure by

- $Y(\hat{\xi}_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \frac{\hat{\xi}_n}{z^{n+1}}$

- $Y(\hat{\xi}^1_{-n_1} \cdots \hat{\xi}^k_{-n_k}|0\rangle, z) = :\prod_{j=1}^{k} \frac{\partial_z^{n_j-1}}{(n_j-1)!} Y(\hat{\xi}^j_{-n_j}|0\rangle, z):$

  $\xi^1, \ldots, \xi^k \in \mathfrak{h}$

  : normal ordered product put the positive modes ($k_i > 0$) to the left, negative modes to the right

- $|\omega\rangle = \frac{1}{2}\sum_{i,j} \chi^{ij}\hat{x}_{-1}\hat{x}_{-1}|0\rangle$ $\quad \chi^{ij}$ orthonormal basis for $\mathfrak{h} = \mathbb{C}^\infty$

- $W(\mathfrak{gl})$ sub VOA of $\mathrm{Sym}(\hat{\mathfrak{h}}_-)|0\rangle$ generated by $E_{+}\Theta_{x_1^1}(\chi_1^1, \chi_2^1)|0\rangle$

  e_i = elementary symmetric polynomial in i variables

$Y(E_{i,1}, z) = W^i(z) = \sum_{k \in \mathbb{Z}} \frac{(W_k^i)}{z^{i+k}}$ ←

Central charge $c = \Gamma$

$V \xrightarrow{L_0} V_0^{\oplus n}$ $\quad \mathrm{Aut}(V_0^{\oplus n}) \ni \exp(2\pi\alpha) \cdot \mathrm{id} \in V_0^{\oplus n}$ always respect the chiral condition ②

$\mathfrak{gl}_\infty \to$ ①

$x_i \to x_i$

$x_i \to -\partial_{x_i}$ $\quad x_i \cdot x_j$

$\exp(\sum \frac{\partial_{x_i^k}}{i+k} \hat{x}_{-i}^k \partial_{x_{i+k}^k})$ preserve ③

generators of Lie algebra

$\exp(a\partial_x)$ translation of variables

$\hookrightarrow \exp(\sum \frac{\partial_{x_i^k}}{i+k} \partial_{x_{i+k}^k})$: "change of parameters" $x_i \to x_i + \frac{\partial_{x_i^k}}{i+k}$

"Riemann Grassmann group"

$L \cdot e^F = 0$

$W^i L(e^F)$

e^F